

# POSITIVE SOLUTIONS OF NONLINEAR THREE-POINT INTEGRAL BOUNDARY VALUE PROBLEMS FOR SECOND-ORDER DIFFERENTIAL EQUATIONS

FAOUZI HADDOUCHI, SLIMANE BENAICHA

ABSTRACT. We investigate the existence of positive solutions to the nonlinear second-order three-point integral boundary value problem

$$\begin{aligned} u''(t) + a(t)f(u(t)) &= 0, \quad 0 < t < T, \\ u(0) &= \beta u(\eta), \quad u(T) = \alpha \int_0^\eta u(s)ds, \end{aligned}$$

where  $0 < \eta < T$ ,  $0 < \alpha < \frac{2T}{\eta^2}$ ,  $0 \leq \beta < \frac{2T - \alpha\eta^2}{\alpha\eta^2 - 2\eta + 2T}$  are given constants. We show the existence of at least one positive solution if  $f$  is either superlinear or sublinear by applying Krasnoselskii's fixed point theorem in cones.

## 1. INTRODUCTION

The study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by II'in and Moiseev [5]. Then Gupta [2] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by several authors. We refer the reader to [1, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21] and the references therein.

Tariboon and Sitthiwirattam [20] proved the existence of positive solutions for the three-point boundary-value problem with integral condition

$$u''(t) + a(t)f(u(t)) = 0, \quad t \in (0, 1), \quad (1.1)$$

$$u(0) = 0, \quad u(1) = \alpha \int_0^\eta u(s)ds, \quad (1.2)$$

where  $0 < \eta < 1$  and  $0 < \alpha < \frac{2}{\eta^2}$ .

This paper is concerned with the existence of positive solutions of the equation

$$u''(t) + a(t)f(u(t)) = 0, \quad t \in (0, T), \quad (1.3)$$

with the three-point integral boundary condition

$$u(0) = \beta u(\eta), \quad u(T) = \alpha \int_0^\eta u(s)ds, \quad (1.4)$$

where  $\alpha > 0$ ,  $\beta \geq 0$ ,  $\eta \in (0, T)$  are given constants. Clearly if  $\beta = 0$  and  $T = 1$ , then (1.4) reduces to (1.2). The purpose of this paper is to give some results for existence

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for positive solutions to (1.3)-(1.4), assuming that  $0 < \alpha < \frac{2T}{\eta^2}$ ,  $0 \leq \beta < \frac{2T - \alpha\eta^2}{\alpha\eta^2 - 2\eta + 2T}$  and  $f$  is either superlinear or sublinear.

Our results extend and complete those obtained by J. Tariboon and T. Sitthiraththam [20]. On the other hand, we point out that the proof of the last part in the sublinear case ( $f_\infty = 0$ ) of Theorem 3.1 in [20] is not correct since it is based on an inequality which is not true. We give a new proof, which is different from that of Theorem 3.1 in [20], and obtain an extended result.

Set

$$f_0 = \lim_{u \rightarrow 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \rightarrow \infty} \frac{f(u)}{u}. \quad (1.5)$$

Then  $f_0 = 0$  and  $f_\infty = \infty$  correspond to the superlinear case, and  $f_0 = \infty$  and  $f_\infty = 0$  correspond to the sublinear case. By the positive solution of (1.3)-(1.4) we mean that function  $u(t)$  is positive on  $0 < t < T$  and satisfies the problem (1.3)-(1.4).

Throughout this paper, we assume the following hypotheses:

(H1)  $f \in C([0, \infty), [0, \infty))$ .

(H2)  $a \in C([0, T], [0, \infty))$  and there exists  $t_0 \in [\eta, T]$  such that  $a(t_0) > 0$ .

The following theorem (Krasnoselskii's fixed-point theorem), will play an important role in the proof of our main results.

**Theorem 1.1** ([6]). *Let  $E$  be a Banach space, and let  $K \subset E$  be a cone. Assume  $\Omega_1, \Omega_2$  are open bounded subsets of  $E$  with  $0 \in \Omega_1$ ,  $\overline{\Omega}_1 \subset \Omega_2$ , and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

*be a completely continuous operator such that either*

*(i)  $\|Au\| \leq \|u\|$ ,  $u \in K \cap \partial\Omega_1$ , and  $\|Au\| \geq \|u\|$ ,  $u \in K \cap \partial\Omega_2$ ; or*

*(ii)  $\|Au\| \geq \|u\|$ ,  $u \in K \cap \partial\Omega_1$ , and  $\|Au\| \leq \|u\|$ ,  $u \in K \cap \partial\Omega_2$*

*hold. Then  $A$  has a fixed point in  $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ .*

## 2. PRELIMINARIES

To prove the main existence results we will employ several straightforward lemmas. These lemmas are based on the linear boundary-value problem.

**Lemma 2.1.** *Let  $\beta \neq \frac{2T - \alpha\eta^2}{\alpha\eta^2 - 2\eta + 2T}$ . Then for  $y \in C([0, T], \mathbb{R})$ , the problem*

$$u''(t) + y(t) = 0, \quad t \in (0, T), \quad (2.1)$$

$$u(0) = \beta u(\eta), \quad u(T) = \alpha \int_0^\eta u(s) ds, \quad (2.2)$$

*has a unique solution*

$$\begin{aligned} u(t) = & \frac{\beta(2T - \alpha\eta^2) - 2\beta(1 - \alpha\eta)t}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^\eta (\eta - s)y(s) ds \\ & + \frac{\alpha\beta\eta - \alpha(\beta - 1)t}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^\eta (\eta - s)^2 y(s) ds \\ & + \frac{2(\beta - 1)t - 2\beta\eta}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^T (T - s)y(s) ds - \int_0^t (t - s)y(s) ds. \end{aligned}$$

*Proof.* From (2.1), we have

$$u(t) = u(0) + u'(0)t - \int_0^t (t-s)y(s)ds. \quad (2.3)$$

Integrating (2.3) from 0 to  $\eta$ , where  $\eta \in (0, T)$ , we have

$$\begin{aligned} \int_0^\eta u(s)ds &= u(0)\eta + u'(0)\frac{\eta^2}{2} - \int_0^\eta \left( \int_0^\tau (\tau-s)y(s)ds \right) d\tau \\ &= u(0)\eta + u'(0)\frac{\eta^2}{2} - \frac{1}{2} \int_0^\eta (\eta-s)^2 y(s)ds. \end{aligned}$$

Since

$$\begin{aligned} u(T) &= u(0) + u'(0)T - \int_0^T (T-s)y(s)ds \quad \text{and} \\ u(\eta) &= u(0) + u'(0)\eta - \int_0^\eta (\eta-s)y(s)ds. \end{aligned}$$

By (2.2), from  $u(0) = \beta u(\eta)$ , we have

$$(\beta - 1)u(0) + \eta\beta u'(0) = \beta \int_0^\eta (\eta-s)y(s)ds,$$

and from  $u(T) = \alpha \int_0^\eta u(s)ds$ , we have

$$(1 - \alpha\eta)u(0) + (T - \alpha\frac{\eta^2}{2})u'(0) = \int_0^T (T-s)y(s)ds - \frac{\alpha}{2} \int_0^\eta (\eta-s)^2 y(s)ds.$$

Therefore,

$$\begin{aligned} u(0) &= \frac{\beta(2T - \alpha\eta^2)}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^\eta (\eta-s)y(s)ds \\ &\quad - \frac{2\beta\eta}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^T (T-s)y(s)ds \\ &\quad + \frac{\alpha\beta\eta}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^\eta (\eta-s)^2 y(s)ds, \\ u'(0) &= \frac{2(\beta - 1)}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^T (T-s)y(s)ds \\ &\quad - \frac{\alpha(\beta - 1)}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^\eta (\eta-s)^2 y(s)ds \\ &\quad - \frac{2\beta(1 - \alpha\eta)}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^\eta (\eta-s)y(s)ds, \end{aligned}$$

from which it follows that

$$\begin{aligned} u(t) &= \frac{\beta(2T - \alpha\eta^2) - 2\beta(1 - \alpha\eta)t}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^\eta (\eta-s)y(s)ds \\ &\quad + \frac{\alpha\beta\eta - \alpha(\beta - 1)t}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^\eta (\eta-s)^2 y(s)ds \\ &\quad + \frac{2(\beta - 1)t - 2\beta\eta}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^T (T-s)y(s)ds - \int_0^t (t-s)y(s)ds. \end{aligned}$$

□

**Lemma 2.2.** *Let  $0 < \alpha < \frac{2T}{\eta^2}$ ,  $0 \leq \beta < \frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$ . If  $y \in C([0, T], [0, \infty))$ , then the unique solution  $u$  of (2.1)-(2.2) satisfies  $u(t) \geq 0$  for  $t \in [0, T]$ .*

*Proof.* From the fact that  $u''(t) = -y(t) \leq 0$ , we have that the graph of  $u(t)$  is concave down on  $(0, T)$  and

$$\int_0^\eta u(s)ds \geq \frac{\eta}{2}(u(0) + u(\eta)). \quad (2.4)$$

Combining (2.2) with (2.4), we get

$$u(T) \geq \frac{\alpha(\beta+1)\eta}{2}u(\eta). \quad (2.5)$$

Since the graph of  $u$  is concave down, we get

$$\frac{u(\eta) - u(0)}{\eta} \geq \frac{u(T) - u(0)}{T}.$$

Combining this with (2.2) and (2.5), we obtain

$$(1 - \beta)\frac{u(\eta)}{\eta} \geq \frac{\alpha(\beta+1)\eta - 2\beta}{2T}u(\eta).$$

If  $u(0) < 0$ , then  $u(\eta) < 0$ . It implies  $\beta \geq \frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$ , a contradiction to  $\beta < \frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$ .

If  $u(T) < 0$ , then  $u(\eta) < 0$ , and the same contradiction emerges. Thus, it is true that  $u(0) \geq 0$ ,  $u(T) \geq 0$ , together with the concavity of  $u(t)$ , we have  $u(t) \geq 0$  for  $t \in [0, T]$ .  $\square$

**Lemma 2.3.** *Let  $\alpha > \frac{2T}{\eta^2}$ ,  $\beta \geq 0$ . If  $y \in C([0, T], [0, \infty))$ , then the problem (2.1)-(2.2) has no positive solutions.*

*Proof.* Suppose that problem (2.1)-(2.2) has a positive solution  $u$  satisfying  $u(t) \geq 0$ ,  $t \in [0, T]$ .

If  $u(T) > 0$ , then  $\int_0^\eta u(s)ds > 0$ . It implies

$$u(T) = \alpha \int_0^\eta u(s)ds > \frac{2T}{\eta^2} \frac{\eta}{2}(u(0) + u(\eta)) = \frac{T(\beta+1)u(\eta)}{\eta} \geq \frac{T u(\eta)}{\eta},$$

that is

$$\frac{u(T)}{T} > \frac{u(\eta)}{\eta},$$

which is a contradiction to the concavity of  $u$ .

If  $u(T) = 0$ , then  $\int_0^\eta u(s)ds = 0$ , this is  $u(t) \equiv 0$  for all  $t \in [0, \eta]$ . If there exists  $t_0 \in (\eta, T)$  such that  $u(t_0) > 0$ , then  $u(0) = u(\eta) < u(t_0)$ , a violation of the concavity of  $u$ . Therefore, no positive solutions exist.  $\square$

**Lemma 2.4.** *Let  $0 < \alpha < \frac{2T}{\eta^2}$ ,  $0 \leq \beta < \frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$ . If  $y \in C([0, T], [0, \infty))$ , then the unique solution  $u$  of (2.1)-(2.2) satisfies*

$$\min_{t \in [\eta, T]} u(t) \geq \gamma \|u\|, \quad \|u\| = \max_{t \in [0, T]} |u(t)|, \quad (2.6)$$

where

$$\gamma := \min \left\{ \frac{\eta}{T}, \frac{\alpha(\beta+1)\eta^2}{2T}, \frac{\alpha(\beta+1)\eta(T-\eta)}{2T - \alpha(\beta+1)\eta^2} \right\} \in (0, 1). \quad (2.7)$$

*Proof.* We divide the proof into three cases. Set  $u(t_1) = \|u\|$ .

Case 1. If  $\eta \leq t_1 \leq T$  and  $\min_{t \in [\eta, T]} u(t) = u(\eta)$ , then the concavity of  $u$  implies that

$$\frac{u(\eta)}{\eta} \geq \frac{u(t_1)}{t_1} \geq \frac{u(t_1)}{T}.$$

Thus,

$$\min_{t \in [\eta, T]} u(t) \geq \frac{\eta}{T} \|u\|.$$

Case 2. If  $\eta \leq t_1 \leq T$  and  $\min_{t \in [\eta, T]} u(t) = u(T)$ , then (2.2), (2.4) and the concavity of  $u$  implies

$$\begin{aligned} u(T) &= \alpha \int_0^\eta u(s) ds \geq \frac{\alpha(\beta+1)\eta^2}{2} \frac{u(\eta)}{\eta} \\ &\geq \frac{\alpha(\beta+1)\eta^2}{2} \frac{u(t_1)}{t_1} \\ &\geq \frac{\alpha(\beta+1)\eta^2}{2} \frac{u(t_1)}{T}. \end{aligned}$$

This implies that

$$\min_{t \in [\eta, T]} u(t) \geq \frac{\alpha(\beta+1)\eta^2}{2T} \|u\|.$$

Case 3. If  $t_1 \leq \eta < T$ , then  $\min_{t \in [\eta, T]} u(t) = u(T)$ . Using the concavity of  $u$  and (2.2), (2.4), we obtain

$$\begin{aligned} u(t_1) &\leq u(T) + \frac{u(T) - u(\eta)}{T - \eta} (t_1 - T) \\ &\leq u(T) + \frac{u(T) - u(\eta)}{T - \eta} (0 - T) \\ &\leq u(T) \left[ 1 - T \frac{1 - \frac{2}{\alpha(\beta+1)\eta}}{T - \eta} \right] \\ &= \frac{2T - \alpha(\beta+1)\eta^2}{\alpha(\beta+1)\eta(T - \eta)} u(T), \end{aligned}$$

from which it follows that

$$\min_{t \in [\eta, T]} u(t) \geq \frac{\alpha(\beta+1)\eta(T - \eta)}{2T - \alpha(\beta+1)\eta^2} \|u\|.$$

Summing up, we have

$$\min_{t \in [\eta, T]} u(t) \geq \gamma \|u\|,$$

where

$$\gamma := \min \left\{ \frac{\eta}{T}, \frac{\alpha(\beta+1)\eta^2}{2T}, \frac{\alpha(\beta+1)\eta(T - \eta)}{2T - \alpha(\beta+1)\eta^2} \right\}.$$

This completes the proof.  $\square$

## 3. EXISTENCE OF POSITIVE SOLUTIONS

Now we are in the position to establish the main result.

**Theorem 3.1.** *Assume (H1) and (H2) hold, and  $0 < \alpha < \frac{2T}{\eta^2}$ ,  $0 \leq \beta < \frac{2T - \alpha\eta^2}{\alpha\eta^2 - 2\eta + 2T}$ . Then the problem (1.3)-(1.4) has at least one positive solution in the case*  
*(i)  $f_0 = 0$  and  $f_\infty = \infty$  (superlinear), or*  
*(ii)  $f_0 = \infty$  and  $f_\infty = 0$  (sublinear).*

*Proof.* It is known that  $0 < \alpha < \frac{2T}{\eta^2}$ ,  $0 \leq \beta < \frac{2T - \alpha\eta^2}{\alpha\eta^2 - 2\eta + 2T}$ . From Lemma 2.1,  $u$  is a solution to the boundary value problem (1.3)-(1.4) if and only if  $u$  is a fixed point of operator  $A$ , where  $A$  is defined by

$$\begin{aligned} Au(t) = & \frac{\beta(2T - \alpha\eta^2) - 2\beta(1 - \alpha\eta)t}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^\eta (\eta - s)a(s)f(u(s))ds \\ & + \frac{\alpha\beta\eta - \alpha(\beta - 1)t}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^\eta (\eta - s)^2a(s)f(u(s))ds \\ & + \frac{2(\beta - 1)t - 2\beta\eta}{(\alpha\eta^2 - 2T) - \beta(2\eta - \alpha\eta^2 - 2T)} \int_0^T (T - s)a(s)f(u(s))ds \\ & - \int_0^t (t - s)a(s)f(u(s))ds. \end{aligned}$$

Denote

$$K = \left\{ u/u \in C([0, T], \mathbb{R}), u \geq 0, \min_{t \in [\eta, T]} u(t) \geq \gamma \|u\| \right\},$$

where  $\gamma$  is defined in (2.7). It is obvious that  $K$  is a cone in  $C([0, T], \mathbb{R})$ . Moreover, from Lemma 2.2 and Lemma 2.4,  $AK \subset K$ . It is also easy to check that  $A : K \rightarrow K$  is completely continuous.

Superlinear case.  $f_0 = 0$  and  $f_\infty = \infty$ . Since  $f_0 = 0$ , we may choose  $H_1 > 0$  so that  $f(u) \leq \epsilon u$ , for  $0 < u \leq H_1$ , where  $\epsilon > 0$  satisfies

$$\epsilon T \frac{2(\beta + 1) + T^{-1}\beta\eta(\alpha\eta + 2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T - s)a(s)ds \leq 1.$$

Thus, if we let

$$\Omega_1 = \{u \in C([0, T], \mathbb{R}) : \|u\| < H_1\},$$

then, for  $u \in K \cap \partial\Omega_1$ , we get

$$\begin{aligned}
Au(t) &\leq \frac{2\beta(1-\alpha\eta)t - \beta(2T - \alpha\eta^2)}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta (\eta - s)a(s)f(u(s))ds \\
&\quad + \frac{\alpha(\beta - 1)t - \alpha\beta\eta}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta (\eta - s)^2 a(s)f(u(s))ds \\
&\quad + \frac{2\beta\eta - 2(\beta - 1)t}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T - s)a(s)f(u(s))ds \\
&\leq \frac{2\beta T + \alpha\beta\eta^2}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta (\eta - s)a(s)f(u(s))ds \\
&\quad + \frac{\alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta (\eta - s)^2 a(s)f(u(s))ds \\
&\quad + \frac{2\beta\eta + 2T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T - s)a(s)f(u(s))ds \\
&\leq \frac{2T(\beta + 1) + \beta\eta(\alpha\eta + 2)}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T - s)a(s)f(u(s))ds \\
&\quad + \frac{\alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta (\eta - s)^2 a(s)f(u(s))ds \\
&\leq \frac{2T(\beta + 1) + \beta\eta(\alpha\eta + 2)}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T - s)a(s)f(u(s))ds \\
&\quad + \frac{\alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T T(T - s)a(s)f(u(s))ds \\
&= T \frac{2(\beta + 1) + T^{-1}\beta\eta(\alpha\eta + 2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T - s)a(s)f(u(s))ds \\
&\leq \epsilon \|u\| T \frac{2(\beta + 1) + T^{-1}\beta\eta(\alpha\eta + 2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T - s)a(s)ds \\
&\leq \|u\|.
\end{aligned}$$

Thus  $\|Au\| \leq \|u\|$ ,  $u \in K \cap \partial\Omega_1$ .

Further, since  $f_\infty = \infty$ , there exists  $\hat{H}_2 > 0$  such that  $f(u) \geq \rho u$  for  $u \geq \hat{H}_2$ , where  $\rho > 0$  is chosen so that

$$\rho\gamma \frac{2\eta}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_\eta^T (T - s)a(s)ds \geq 1.$$

Let  $H_2 = \max\{2H_1, \frac{\hat{H}_2}{\gamma}\}$  and  $\Omega_2 = \{u \in C([0, T], \mathbb{R}) : \|u\| < H_2\}$ . Then  $u \in K \cap \partial\Omega_2$  implies that

$$\min_{t \in [\eta, T]} u(t) \geq \gamma \|u\| = \gamma H_2 \geq \hat{H}_2,$$

and so,

$$\begin{aligned}
Au(\eta) &= \frac{2\beta(1-\alpha\eta)\eta - \beta(2T-\alpha\eta^2)}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta (\eta-s)a(s)f(u(s))ds \\
&\quad + \frac{\alpha(\beta-1)\eta - \alpha\beta\eta}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta (\eta-s)^2 a(s)f(u(s))ds \\
&\quad + \frac{2\beta\eta - 2(\beta-1)\eta}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T-s)a(s)f(u(s))ds \\
&\quad - \int_0^\eta (\eta-s)a(s)f(u(s))ds \\
&= \frac{2\eta}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T-s)a(s)f(u(s))ds \\
&\quad - \frac{\alpha\eta}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta (\eta^2 - 2\eta s + s^2)a(s)f(u(s))ds \\
&\quad - \frac{2T-\alpha\eta^2}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta (\eta-s)a(s)f(u(s))ds \\
&= \frac{2\eta}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T-s)a(s)f(u(s))ds \\
&\quad + \frac{\alpha\eta^2}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta sa(s)f(u(s))ds \\
&\quad + \frac{2T}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta sa(s)f(u(s))ds \\
&\quad - \frac{\alpha\eta}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta s^2 a(s)f(u(s))ds \\
&\quad - \frac{2T\eta}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta a(s)f(u(s))ds \\
&= \frac{2\eta}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_\eta^T (T-s)a(s)f(u(s))ds \\
&\quad + \frac{2(T-\eta)}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta sa(s)f(u(s))ds \\
&\quad + \frac{\alpha\eta}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta s(\eta-s)a(s)f(u(s))ds \\
&\geq \frac{2\eta}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_\eta^T (T-s)a(s)f(u(s))ds \\
&\geq \frac{2\eta\rho}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_\eta^T (T-s)a(s)u(s)ds \\
&\geq \frac{2\eta\rho\gamma\|u\|}{(2T-\alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_\eta^T (T-s)a(s)ds \\
&\geq \|u\|.
\end{aligned}$$

Hence,  $\|Au\| \geq \|u\|$ ,  $u \in K \cap \partial\Omega_2$ . By the first part of Theorem 1.1,  $A$  has a fixed point in  $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$  such that  $H_1 \leq \|u\| \leq H_2$ . This completes the superlinear part of the theorem.



Sublinear case.  $f_0 = \infty$  and  $f_\infty = 0$ . Since  $f_0 = \infty$ , choose  $H_3 > 0$  such that  $f(u) \geq Mu$  for  $0 < u \leq H_3$ , where  $M > 0$  satisfies

$$M\gamma \frac{2\eta}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_\eta^T (T-s)a(s)ds \geq 1.$$

Let  $\Omega_3 = \{u \in C([0, T], \mathbb{R}) : \|u\| < H_3\}$ , then for  $u \in K \cap \partial\Omega_3$ , we get

$$\begin{aligned} Au(\eta) &= \frac{2\eta}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T-s)a(s)f(u(s))ds \\ &\quad - \frac{\alpha\eta}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta (\eta-s)^2 a(s)f(u(s))ds \\ &\quad - \frac{2T - \alpha\eta^2}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta (\eta-s)a(s)f(u(s))ds \\ &\geq \frac{2\eta}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_\eta^T (T-s)a(s)f(u(s))ds \\ &\geq \frac{2\eta M}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_\eta^T (T-s)a(s)u(s)ds \\ &\geq M\gamma \frac{2\eta\|u\|}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_\eta^T (T-s)a(s)ds \\ &\geq \|u\|. \end{aligned}$$

Thus  $\|Au\| \geq \|u\|$ ,  $u \in K \cap \partial\Omega_3$ . Now, since  $f_\infty = 0$ , there exists  $\hat{H}_4 > 0$  so that  $f(u) \leq \lambda u$  for  $u \geq \hat{H}_4$ , where  $\lambda > 0$  satisfies

$$\lambda T \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T-s)a(s)ds \leq 1.$$

We consider two cases:

Case (i). Suppose  $f$  is bounded, say  $f(u) \leq N$  for all  $u \in [0, \infty)$ . Choosing  $H_4 \geq \max\{2H_3, \frac{N}{\lambda}\}$ . For  $u \in K$  with  $\|u\| = H_4$ , we have

$$\begin{aligned} Au(t) &= \frac{2\beta(1 - \alpha\eta)t - \beta(2T - \alpha\eta^2)}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta (\eta-s)a(s)f(u(s))ds \\ &\quad + \frac{\alpha(\beta-1)t - \alpha\beta\eta}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^\eta (\eta-s)^2 a(s)f(u(s))ds \\ &\quad + \frac{2\beta\eta - 2(\beta-1)t}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T-s)a(s)f(u(s))ds \\ &\quad - \int_0^t (t-s)a(s)f(u(s))ds \\ &\leq T \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T-s)a(s)f(u(s))ds \\ &\leq NT \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T-s)a(s)ds \\ &\leq H_4 \lambda T \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T-s)a(s)ds \\ &\leq H_4, \end{aligned}$$

and therefore  $\|Au\| \leq \|u\|$ .  $\square$

Case (ii). If  $f$  is unbounded, then we know from  $f \in C([0, \infty), [0, \infty))$  that there is  $H_4$ :  $H_4 \geq \max\{2H_3, \frac{\bar{H}_4}{\gamma}\}$  such that

$$f(u) \leq f(H_4) \text{ for } u \in [0, H_4].$$

Then for  $u \in K$  and  $\|u\| = H_4$ , we have

$$\begin{aligned} Au(t) &\leq T \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T-s)a(s)f(u(s))ds \\ &\leq T \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T-s)a(s)f(H_4)ds \\ &\leq H_4 \lambda T \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T (T-s)a(s)ds \\ &\leq H_4 = \|u\|. \end{aligned}$$

Therefore, in either case we may set

$$\Omega_4 = \{u \in C([0, T], \mathbb{R}) : \|u\| < H_4\},$$

and for  $u \in K \cap \partial\Omega_4$  we may have  $\|Au\| \leq \|u\|$ . By the second part of Theorem 1.1, it follows that  $A$  has a fixed point in  $K \cap (\bar{\Omega}_4 \setminus \Omega_3)$  such that  $H_3 \leq \|u\| \leq H_4$ . This completes the sublinear part of the theorem. Therefore, the problem (1.3)-(1.4) has at least one positive solution.

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FAOUZI HADDOUCHI, DEPARTMENT OF PHYSICS, UNIVERSITY OF SCIENCES AND TECHNOLOGY OF ORAN, EL MNAOUAR, BP 1505, 31000 ORAN, ALGERIA  
*E-mail address:* fhaddouchi@gmail.com

SLIMANE BENAICHA, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ORAN, ES-SENIA, 31000 ORAN, ALGERIA  
*E-mail address:* slimanebenaicha@yahoo.fr